Analysis of an uncertain volatility model.

Numerical methods

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Asian Options

- Geometric Average Asian options PDE:
  - $\partial_t U + rS\partial_S U + \frac{1}{2}\sigma^2 S^2 \partial_{SS} U + \log(S)\partial_A U = rU,$
  - S follows a standard geometric Brownian motion with volatility σ.
  - U price of a geometric average Asian option
  - A path-dependent variable
  - r risk-free rate
- By change of variables:
  - $\mathcal{K}u := \partial_{xx}u + x\partial_y u - \partial_t u = 0$
  - strongly degenerate due to the lack of diffusion in the y-direction
  - An *explicit* fundamental solution of Gaussian type exists (it is C^∞ outside the diagonal)

Hobson-Rogers model

- $dS_t = \mu(D_t)S_t dt + \sigma(D_t)S_t dB_t,$
- $D_t = Z_t - \int_0^{+\infty} \lambda e^{-\lambda\tau} Z_{t-\tau} d\tau, \quad Z_t = \log(e^{-rt}S_t)$
- D_t deviation from the trend
- HR propose $\sigma(d) = \max\left(\eta\sqrt{1+\varepsilon d^2}, \sigma_{max}\right)$
- No exogenous source of risk has been included
- Usual no-arbitrage arguments provide unique option prices
- Allows for smiles

Smile patterns

Smile patterns

HR models and Kolmogorov PDEs

- Price of contingent claim f satisfies the PDE

$$\frac{\sigma^2(D)}{2}\left(S^2\partial_{SS}f + \partial_{DD}f + 2S\partial_{DS}f - \partial_D f\right) + rS\partial_S f - \lambda D\partial_D f - \partial_t f = rf$$

- It is a degenerate PDE: the quadratic form is $\frac{\sigma^2}{2}\begin{pmatrix} S^2 & S \\ S & 1 \end{pmatrix}$
- PDE of Kolmogorov or Ornstein-Uhlenbeck type
- By change of variables:

$$a\,(\partial_{xx}u - \partial_x u) + x\partial_y u - \partial_\tau u = 0$$

- where $t = -\log(1-\tau)/\lambda$, $S = Ke^{x-rt}$, $D = x - y/(1-\tau)$ and $a(x,y,\tau) = \frac{\sigma^2\left(x - \frac{y}{1-\tau}\right)}{2\lambda(1-\tau)}$
- Main directional derivatives: ∂_x and $Y = x\partial_y - \partial_t$

Euclidean Scheme

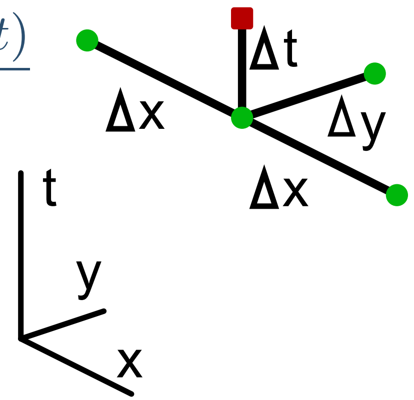
- Approximate:

$$\partial_{xx}u \simeq \frac{u(x+\Delta_x,y,t)-2u(x,y,t)+u(x-\Delta_x,y,t)}{\Delta_x^2}$$

$$\partial_{x}u \simeq \frac{u(x+\Delta_x,y,t)-u(x-\Delta_x,y,t)}{2\Delta_x}$$

$$\partial_{y}u \simeq \frac{u(x,y,t)-u(x,y-\mathrm{sign}(y)\Delta_y,t)}{\mathrm{sign}(y)\Delta_y}$$

$$\partial_{t}u \simeq \frac{u(x,y,t+\Delta_t)-u(x,y,t)}{\Delta_t}$$



- Errors:
  - $\|\mathcal{K}u - \mathcal{K}_E u\|_{L^\infty} \leq C(\Delta_y + \Delta_t + \Delta_x^2)$
  - C depends on a, $\partial_x^3 u$, $\partial_x^4 u$, $x\partial_y^2 u$ and $\partial_t^2 u$

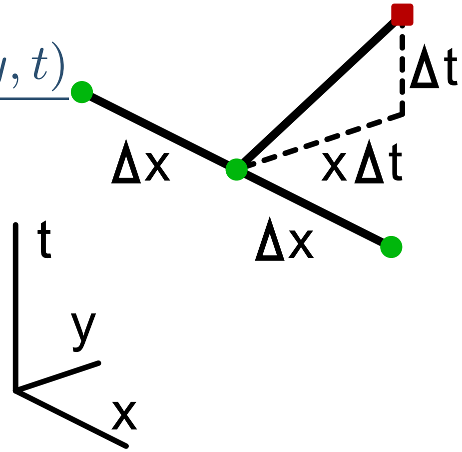
Kolmogorov Scheme

- Approximate:

$$\partial_{xx}u \simeq \frac{u(x+\Delta_x,y,t)-2u(x,y,t)+u(x-\Delta_x,y,t)}{\Delta_x^2}$$

$$\partial_x u \simeq \frac{u(x+\Delta_x,y,t)-u(x-\Delta_x,y,t)}{2\Delta_x}$$

$$Yu \simeq \frac{u(x,y,t)-u(x,y-x\Delta_t,t+\Delta_t)}{\Delta_t}$$



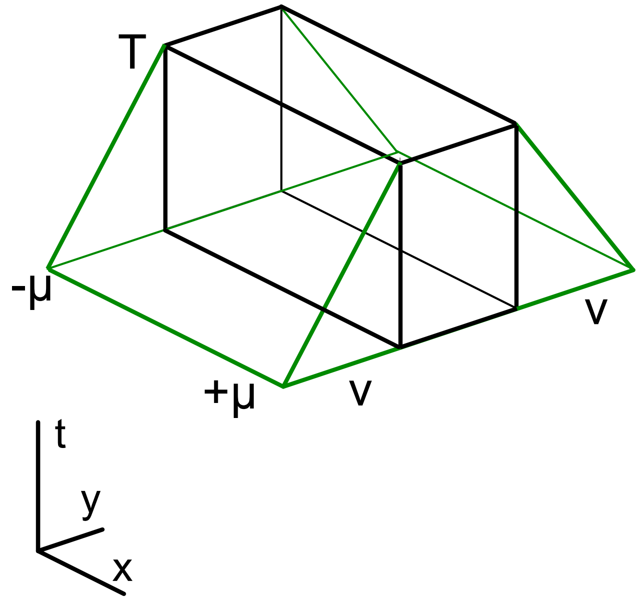
- Requires $\Delta_y = \Delta_x \Delta_t$
- Errors:
  - $\|\mathcal{K}u - \mathcal{K}_K u\|_{L^\infty} \leq C(\Delta_t + \Delta_x^2)$
  - C depends on $\partial_x^3 u$, $\partial_x^4 u$ and $Y^2 u$
- Interpolation introduces an additional error of $O\left(\Delta_y^2/\Delta_t \|\partial_y^2 u\|\right)$

Border Conditions

- Usual border conditions hold as $x \longrightarrow \pm\infty$.

Domain of Dependence

- Black box: region of interest
- Green Trapezoid: domain of dependence
- Kolmogorov Numerical Domain = Analytical Domain
  - $v = T\mu$
- Euclidean Numerical Domain
  - $v = N\Delta_y = T\Delta_y/\Delta_t$



Digital Options - Low Volatility

	Euclidean			Kolmogorov		
l	Price	Err %	Exec. time	price	Err %	Exec. time
50	0.5520	4.56	0.2	0.4460	15.51	4.9
100	0.5116	3.08	1.6	0.4879	7.57	19.5
150	0.4983	5.60	7.0	0.5013	5.03	43.8
200	0.4917	6.85	19.6	0.5081	3.75	81.8
250	0.4878	7.59	93.7	0.5121	2.99	134.3

- At the money $S_0 = K$, $T = 0.25$, $D_0 = 0.1$ and $\eta = 0.2$.
- Monte carlo: 0.5279

Digital Options - High Volatility

I	Euclidean			Kolmogorov		
	Price	Err %	Exec. time	Price	Err %	Exec. time
50	0.4678	28.60	5.1	0.6409	2.18	5.1
100	0.4640	29.18	60.6	0.6485	1.02	20.8
150	0.4628	29.36	243.5	0.6510	0.64	47.2
200	0.4622	29.45	706.7	0.6522	0.45	84.9
250	0.4619	29.50	1635.5	0.6530	0.33	148.5

- At the money $S_0 = K$, $T = 0.75$, $D_0 = 0.1$ and $\eta = 0.7$.
- Monte carlo: 0.6552

European Options - Low Vol.

	Euclidean			Kolmogorov		
I	Price	Err %	Exec. time	Price	Err %	Exec. time
50	0.0406	0.00	0.2	0.0399	1.72	4.9
100	0.0409	0.73	1.6	0.0406	0.00	19.5
150	0.0409	0.73	7.0	0.0406	0.00	43.8
200	0.0409	0.73	19.6	0.0407	0.24	81.8
250	0.0409	0.73	93.7	0.0407	0.24	134.3

- At the money $S_0 = K$, $T = 0.25$, $D_0 = 0.1$ and $\eta = 0.2$.
- Monte carlo: 0.0406

European Options - High Vol.

	Euclidean			Kolmogorov		
I	Price	Err %	Exec. time	Price	Err %	Exec. time
50	0.2699	1.62	5.1	0.2657	0.03	4.9
100	0.2690	1.28	52.2	0.2658	0.07	19.5
150	0.2687	1.17	230.8	0.2659	0.11	43.8
200	0.2686	1.13	706.7	0.2659	0.11	81.8
250	0.2685	1.09	1635.5	0.2659	0.11	134.3

- At the money $S_0 = K$, $T = 0.75$, $D_0 = 0.1$ and $\eta = 0.7$.
- Monte carlo: 0.2656

Calibration Problem

- The Kolmogorov PDE can be discretized as

$$(\bar{Y} - \bar{A}\bar{D})\bar{u} = \bar{u}_b$$

  - $\bar{u}$: vector containing the $u(x, y, t)$ at the grid points
  - $\bar{Y}\bar{u}$: discretization of the operator $Yu = \partial_t - x\partial_y$
  - $\bar{D}\bar{u}$: discretization of the operator $\partial_{xx} - \partial_x$
  - $\bar{A}$: diagonal matrix containing the values of a at the grid points
  - $\bar{u}_b$: vector with initial/border values of u
- If $\sigma(s) = \sum_i \alpha_i s_i(s)$, then $\bar{A} = \sum_i \alpha_i A_i$.

Calibration Problem

- u should be fitted to market prices:

$$B\bar{u} + \varepsilon = b$$

  - b: vector of observed prices
  - B: interpolation matrix, each row interpolate to the ith predicted price
  - ε: disturbances. For simplicity ε are i.i.d.
- PDE constrained optimization problem

$$\begin{aligned} \min_{u,\alpha} \quad & \varepsilon^T\varepsilon + \rho\alpha^T\alpha \\ \text{s.t.} \quad & (\bar{Y} - \sum_i \alpha_i \bar{A}_i \bar{D})\bar{u} = \bar{u}_b, \\ \text{and} \quad & B\bar{u} + \varepsilon = b \end{aligned}$$