Algorithms for solving SUR models: a comparative study

Paolo Foschi, David Belsley and Erricos J. Kontoghiorghes

Contents

- The SUR model.
- The GLS estimator for the SUR model
- The GLLSP formulation of the SUR model
- Recursive solution
- Reducing the size of the SUR model
- Computational comparison

Seemingly Unrelated Regressions (SUR) model

- $y_i = X_i\beta_i + u_i, \quad i = 1, 2, \ldots, G.$
- $y_i, u_i : T \times 1, \quad X_i : T \times k_i$ and $\beta_i : k_i \times 1.$
- $E(u_i) = 0$ and $E(u_i u_j^T) = \sigma_{ij} I_T.$
- A compact form of the SUR model is

$$\begin{pmatrix} y_1 \\ \vdots \\ y_G \end{pmatrix} = \begin{pmatrix} X_1 & & \\ & \ddots & \\ & & X_G \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_G \end{pmatrix} + \begin{pmatrix} u_1 \\ \vdots \\ u_G \end{pmatrix}$$

or

$$\text{vec}(Y) = (\oplus_i X_i)\, \text{vec}(\{\beta_i\}) + \text{vec}(U).$$

- $Y = (y_1 \cdots y_G), \quad U = (u_1 \cdots u_G)$ and $\text{vec}(U) \sim (0, \Sigma \otimes I_T).$

Best Linear Unbiased Estimators

- The Generalized Least Squares (GLS) estimator derives from

$$\underset{\beta_1,\ldots,\beta_G}{\text{argmin}} \left\| (C^{-1} \otimes I_T)\Big(\text{vec}(Y) - (\oplus_i X_i)\Big)\text{vec}(\{\beta_i\})) \right\|_2$$

- $\Sigma = CC^T$

- The GLS is given by

$$\text{vec}(\{\hat{\beta}_i\}) = \Big((\oplus_i X_i^T)(\Sigma^{-1} \otimes I_T)(\oplus_i X_i)\Big)^{-1} (\oplus_i X_i^T)\, \text{vec}(Y\Sigma^{-1})$$

- Fails when Σ is ill-conditioned

Computing the GLS estimator

- Compute the QR decomposition (QRD) of $(C^{-1} \otimes I_T)(\oplus_i X_i)$

1. Compute the QRDs $\quad Q_i^T X_i = \begin{pmatrix} R_i \\ 0 \end{pmatrix} \begin{matrix} k_i \\ T-k_i \end{matrix}, \quad Q_i^T = \begin{pmatrix} \widetilde{Q}_i^T \\ \widehat{Q}_i^T \end{pmatrix} \begin{matrix} k_i \\ T-k_i \end{matrix}$

2. Compute

$$Q^T \left((C^{-1} \otimes I_T)(\oplus_i X_i)\right) = \begin{pmatrix} \widetilde{W} \\ \widehat{W} \end{pmatrix}, \quad Q^T = \begin{pmatrix} \oplus_i \widetilde{Q}_i^T \\ \oplus_i \widehat{Q}_i^T \end{pmatrix} \begin{matrix} K \\ GT-K \end{matrix}$$

3. Compute the updating QRD $\quad P^T \begin{pmatrix} \widetilde{W} \\ \widehat{W} \end{pmatrix} = \begin{pmatrix} \overline{R} \\ 0 \end{pmatrix}$

- $\widetilde{W}_{ij} = \gamma_{ij} \widetilde{Q}_i^T X_j, \quad \widehat{W}_{ij} = \gamma_{ij} \widehat{Q}_i^T X_j$ and $C^{-1} = [\gamma_{ij}]$

- $\widetilde{W}$ is upper triangular and $\widehat{W}$ is strictly block upper-triangular

Diagonally-based strategy

$$\left(\begin{array}{cccc} \widetilde{W}_{1,1} & \widetilde{W}_{1,2} & \widetilde{W}_{1,3} & \widetilde{W}_{1,4} \\ 0 & \widetilde{W}_{2,2} & \widetilde{W}_{2,3} & \widetilde{W}_{2,4} \\ 0 & 0 & \widetilde{W}_{3,3} & \widetilde{W}_{3,4} \\ 0 & 0 & 0 & \widetilde{W}_{4,4} \\ \hline 0 & \widehat{W}_{1,2} & \widehat{W}_{1,3} & \widehat{W}_{1,4} \\ 0 & 0 & \widehat{W}_{2,3} & \widehat{W}_{2,4} \\ 0 & 0 & 0 & \widehat{W}_{3,4} \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{cccc} \widetilde{W}_{1,1} & \widetilde{W}_{1,2} & \widetilde{W}_{1,3} & \widetilde{W}_{1,4} \\ 0 & \widetilde{W}_{2,2} & \widetilde{W}_{2,3} & \widetilde{W}_{2,4} \\ 0 & 0 & \widetilde{W}_{3,3} & \widetilde{W}_{3,4} \\ 0 & 0 & 0 & \widetilde{W}_{4,4} \\ \hline 0 & 0 & \widehat{W}_{1,3} & \widehat{W}_{1,4} \\ 0 & 0 & 0 & \widehat{W}_{2,4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc} \widetilde{W}_{1,1} & \widetilde{W}_{1,2} & \widetilde{W}_{1,3} & \widetilde{W}_{1,4} \\ 0 & \widetilde{W}_{2,2} & \widetilde{W}_{2,3} & \widetilde{W}_{2,4} \\ 0 & 0 & \widetilde{W}_{3,3} & \widetilde{W}_{3,4} \\ 0 & 0 & 0 & \widetilde{W}_{4,4} \\ \hline 0 & 0 & 0 & \widehat{W}_{1,4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{cccc} \widetilde{W}_{1,1} & \widetilde{W}_{1,2} & \widetilde{W}_{1,3} & \widetilde{W}_{1,4} \\ 0 & \widetilde{W}_{2,2} & \widetilde{W}_{2,3} & \widetilde{W}_{2,4} \\ 0 & 0 & \widetilde{W}_{3,3} & \widetilde{W}_{3,4} \\ 0 & 0 & 0 & \widetilde{W}_{4,4} \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Column-based strategy

$$\begin{pmatrix} \widetilde{W}_{1,1} & \widetilde{W}_{1,2} & \widetilde{W}_{1,3} & \widetilde{W}_{1,4} \\ 0 & \widetilde{W}_{2,2} & \widetilde{W}_{2,3} & \widetilde{W}_{2,4} \\ 0 & 0 & \widetilde{W}_{3,3} & \widetilde{W}_{3,4} \\ 0 & 0 & 0 & \widetilde{W}_{4,4} \\ \hline 0 & \widehat{W}_{1,2} & \widehat{W}_{1,3} & \widehat{W}_{1,4} \\ 0 & 0 & \widehat{W}_{2,3} & \widehat{W}_{2,4} \\ 0 & 0 & 0 & \widehat{W}_{3,4} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \widetilde{W}_{1,1} & \widetilde{W}_{1,2} & \widetilde{W}_{1,3} & \widetilde{W}_{1,4} \\ 0 & \widetilde{W}_{2,2} & \widetilde{W}_{2,3} & \widetilde{W}_{2,4} \\ 0 & 0 & \widetilde{W}_{3,3} & \widetilde{W}_{3,4} \\ 0 & 0 & 0 & \widetilde{W}_{4,4} \\ \hline 0 & 0 & \widehat{W}_{1,3} & \widehat{W}_{1,4} \\ 0 & 0 & \widehat{W}_{2,3} & \widehat{W}_{2,4} \\ 0 & 0 & 0 & \widehat{W}_{3,4} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \widetilde{W}_{1,1} & \widetilde{W}_{1,2} & \widetilde{W}_{1,3} & \widetilde{W}_{1,4} \\ 0 & \widetilde{W}_{2,2} & \widetilde{W}_{2,3} & \widetilde{W}_{2,4} \\ 0 & 0 & \widetilde{W}_{3,3} & \widetilde{W}_{3,4} \\ 0 & 0 & 0 & \widetilde{W}_{4,4} \\ \hline 0 & 0 & 0 & \widehat{W}_{1,4} \\ 0 & 0 & 0 & \widehat{W}_{2,4} \\ 0 & 0 & 0 & \widehat{W}_{3,4} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \widetilde{W}_{1,1} & \widetilde{W}_{1,2} & \widetilde{W}_{1,3} & \widetilde{W}_{1,4} \\ 0 & \widetilde{W}_{2,2} & \widetilde{W}_{2,3} & \widetilde{W}_{2,4} \\ 0 & 0 & \widetilde{W}_{3,3} & \widetilde{W}_{3,4} \\ 0 & 0 & 0 & \widetilde{W}_{4,4} \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Generalized Linear Least Squares Problem (GLLSP)

- Solve the GLLSP

$$\underset{V,\beta_1,\ldots,\beta_G}{\text{argmin}} \; \|V\|_F^2 \quad \text{s.t.} \quad \text{vec}(Y) = (\oplus_i X_i)\,\text{vec}(\{\beta_i\}) + (C \otimes I)\,\text{vec}(V).$$

- $\Sigma = CC^T$, where $C : G \times g$ and $\text{rank}(\Sigma) = g \leq G$.

- $V : T \times g$, $VC^T = U$ and $\text{vec}(V) \sim (0, I_{gT})$.

- Compute the Best Linear Unbiased Estimator (BLUE) of the SUR model.

- Do not require that Σ is non-singular.

The Generalized QR decomposition and GLLSP

- Compute the Generalized QR Decomposition (GQRD):

$$Q^T(\oplus_i X_i) = \begin{pmatrix} \oplus_i R_i \\ 0 \end{pmatrix} \begin{matrix} {\scriptstyle K} \\ {\scriptstyle T-K} \end{matrix} \quad \text{and} \quad Q^T(C \otimes I_T)P = \begin{pmatrix} W_{11} & W_{12} \\ 0 & W_{22} \end{pmatrix} \begin{matrix} {\scriptstyle K} \\ {\scriptstyle T-K} \end{matrix}$$

- Re-write the GLLSP as

$$\underset{V,\beta_1,\ldots,\beta_G}{\text{argmin}} \left\| P^T \operatorname{vec}(V) \right\|^2 \quad \text{s.t.}$$

$$Q^T \operatorname{vec}(Y) = Q^T(\oplus_i X_i)\operatorname{vec}(\{\beta_i\}) + Q^T(C \otimes I)PP^T \operatorname{vec}(V).$$

- Let $Q^T \operatorname{vec}(Y) = \begin{pmatrix} \operatorname{vec}(\{\tilde{y}_i\}) \\ \operatorname{vec}(\{\hat{y}_i\}) \end{pmatrix}$ and $P^T \operatorname{vec}(V) = \begin{pmatrix} \operatorname{vec}(\{\tilde{v}_i\}) \\ \operatorname{vec}(\{\hat{v}_i\}) \end{pmatrix}$.

Solution of the GLLSP

- Write the GLLSP as

$$\underset{V,\beta_1,\ldots,\beta_G}{\text{argmin}} \sum_{i=1}^{G} \left(\|\tilde{v}_i\|^2 + \|\hat{v}_i\|^2 \right) \quad \text{s.t.}$$

$$\begin{pmatrix} \text{vec}(\{\tilde{y}_i\}) \\ \text{vec}(\{\hat{y}_i\}) \end{pmatrix} = \begin{pmatrix} \oplus_i R_i \\ 0 \end{pmatrix} \text{vec}(\{\beta_i\}) + \begin{pmatrix} W_{11} & W_{12} \\ 0 & W_{22} \end{pmatrix} \begin{pmatrix} \text{vec}(\{\tilde{v}_i\}) \\ \text{vec}(\{\hat{v}_i\}) \end{pmatrix}$$

- Solution:

1. Set $\text{vec}(\{\tilde{v}_i\}) = 0$.
2. Solve the triangular system

$$\begin{pmatrix} \oplus_i R_i & W_{12} \\ 0 & W_{22} \end{pmatrix} \begin{pmatrix} \text{vec}(\{\beta_i\}) \\ \text{vec}(\{\hat{v}_i\}) \end{pmatrix} = \begin{pmatrix} \text{vec}(\{\tilde{y}_i\}) \\ \text{vec}(\{\hat{y}_i\}) \end{pmatrix}.$$

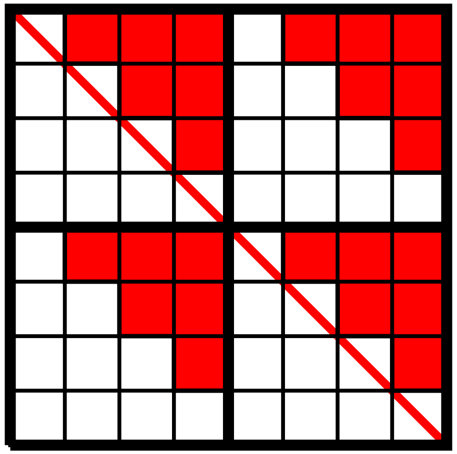
Computation of the RQD of $Q^T(C \otimes I_T)$

- $Q^T(C \otimes I_T)P = \begin{pmatrix} W_{AA} & W_{AB} \\ 0 & W_{BB} \end{pmatrix}$
- $Q^T(C \otimes I_T)Q = \begin{pmatrix} \widetilde{W}_{AA} & \widetilde{W}_{AB} \\ \widetilde{W}_{BA} & \widetilde{W}_{BB} \end{pmatrix}$
- $\widetilde{W}_{AA}$ and $\widetilde{W}_{BB}$ are upper triangular
- $\widetilde{W}_{AB}$ and $\widetilde{W}_{BA}$ are strictly block upper triangular

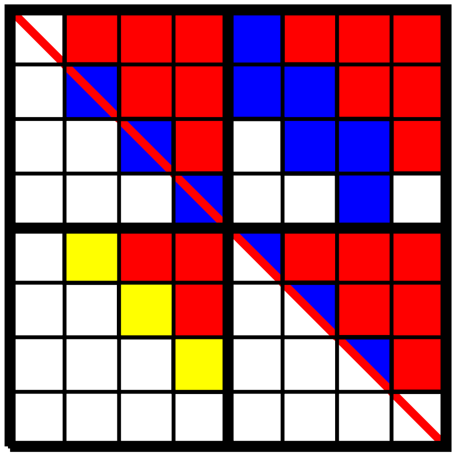
1. Compute the updating RQD $\begin{pmatrix} \widetilde{W}_{BA} & \widetilde{W}_{BB} \end{pmatrix} \widetilde{P} = \begin{pmatrix} 0 & W_{BB} \end{pmatrix}$
2. Compute $\begin{pmatrix} \widetilde{W}_{AA} & \widetilde{W}_{AB} \end{pmatrix} \widetilde{P} = \begin{pmatrix} W_{AA} & W_{AB} \end{pmatrix}$

Diagonally-based strategy

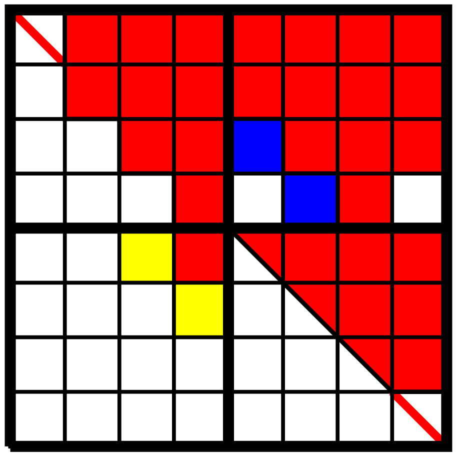
Initial matrix



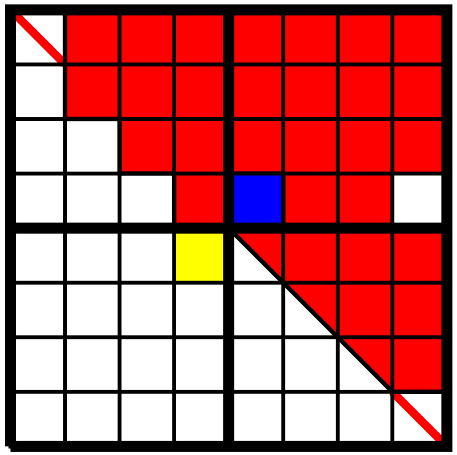
First Step



Second Step



Third Step



Interleaving approach: basic idea

- Consider the GLLSP: $\text{argmin}_{v,\beta} \|v\|$ **s.t.** $y = X\beta + Cv$

- $Q^T \overset{\begin{matrix} k & 1 \end{matrix}}{\begin{pmatrix} X & y \end{pmatrix}} = \overset{\begin{matrix} k & 1 \end{matrix}}{\begin{pmatrix} X_1 & y_1 \\ 0 & y_2 \end{pmatrix}} \begin{matrix} S \\ T-S \end{matrix}$, $Q^T C P = \overset{\begin{matrix} S & T-S \end{matrix}}{\begin{pmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{pmatrix}} \begin{matrix} S \\ T-S \end{matrix}$

- The GLLSP can be rewritten as

$$\underset{v,\beta}{\text{argmin}} \|v_1\|^2 + \|v_2\|^2 \quad \textbf{s.t.} \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ 0 \end{pmatrix} \beta + \begin{pmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

which is equivalent to

$$\underset{v,\beta}{\text{argmin}} \|v_1\|^2 \quad \textbf{s.t.} \quad y_1 - C_{12} C_{22}^{-1} y_2 = X_1 \beta + C_{11} v_1$$

- This process is iteratively applied until $S = k$

Interleaving approach for the SUR model

- Assume $k_1 = \cdots = k_G = k$

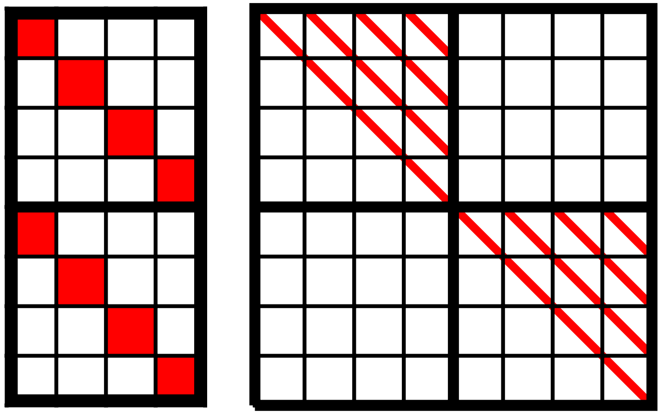
- Partition $X_i = \begin{pmatrix} X_{i1} \\ X_{i2} \end{pmatrix} \begin{matrix} k \\ T-k \end{matrix}$ and $y_i = \begin{pmatrix} y_{i1} \\ y_{i2} \end{pmatrix} \begin{matrix} k \\ T-k \end{matrix}$

- Rewrite the GLLSP as

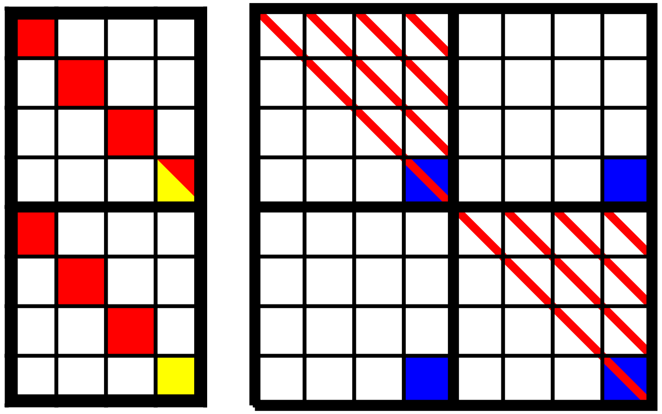
$$\begin{pmatrix} y_{11} \\ \vdots \\ y_{G1} \\ y_{12} \\ \vdots \\ y_{G2} \end{pmatrix} = \begin{pmatrix} X_{11} & & \\ & \ddots & \\ & & X_{G1} \\ X_{12} & & \\ & \ddots & \\ & & X_{G2} \end{pmatrix} \text{vec}(\{\beta_i\}) + \begin{pmatrix} C \otimes I_k & 0 \\ 0 & C \otimes I_k \end{pmatrix} \begin{pmatrix} v_{11} \\ \vdots \\ v_{G1} \\ v_{12} \\ \vdots \\ v_{G2} \end{pmatrix}$$

Interleaving approach for the SUR model

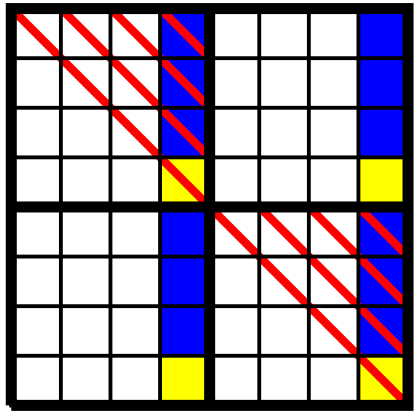
Initial matrices



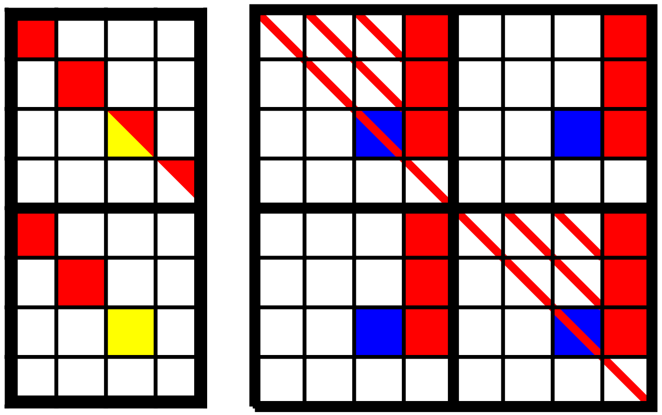
First Step (QRD)



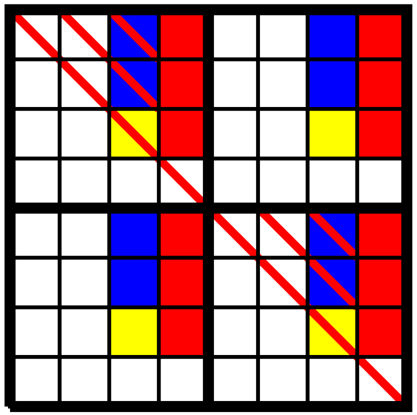
First Step (RQD)



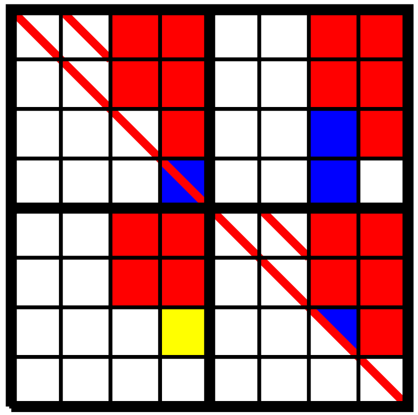
Second Step (QRD)



Second Step (RQD 1)



Second Step (RQD 2)



Recursive Algorithm

- Let $X_i = \begin{pmatrix} X_i^{(1)} \\ X_i^{(2)} \\ \vdots \\ X_i^{(s)} \end{pmatrix} \begin{matrix} T_1 \\ T_2 \\ \vdots \\ T_s \end{matrix}$, $Y = \begin{pmatrix} Y^{(1)} \\ Y^{(2)} \\ \vdots \\ Y^{(s)} \end{pmatrix}$ and $U = \begin{pmatrix} U^{(1)} \\ U^{(2)} \\ \vdots \\ U^{(s)} \end{pmatrix}$

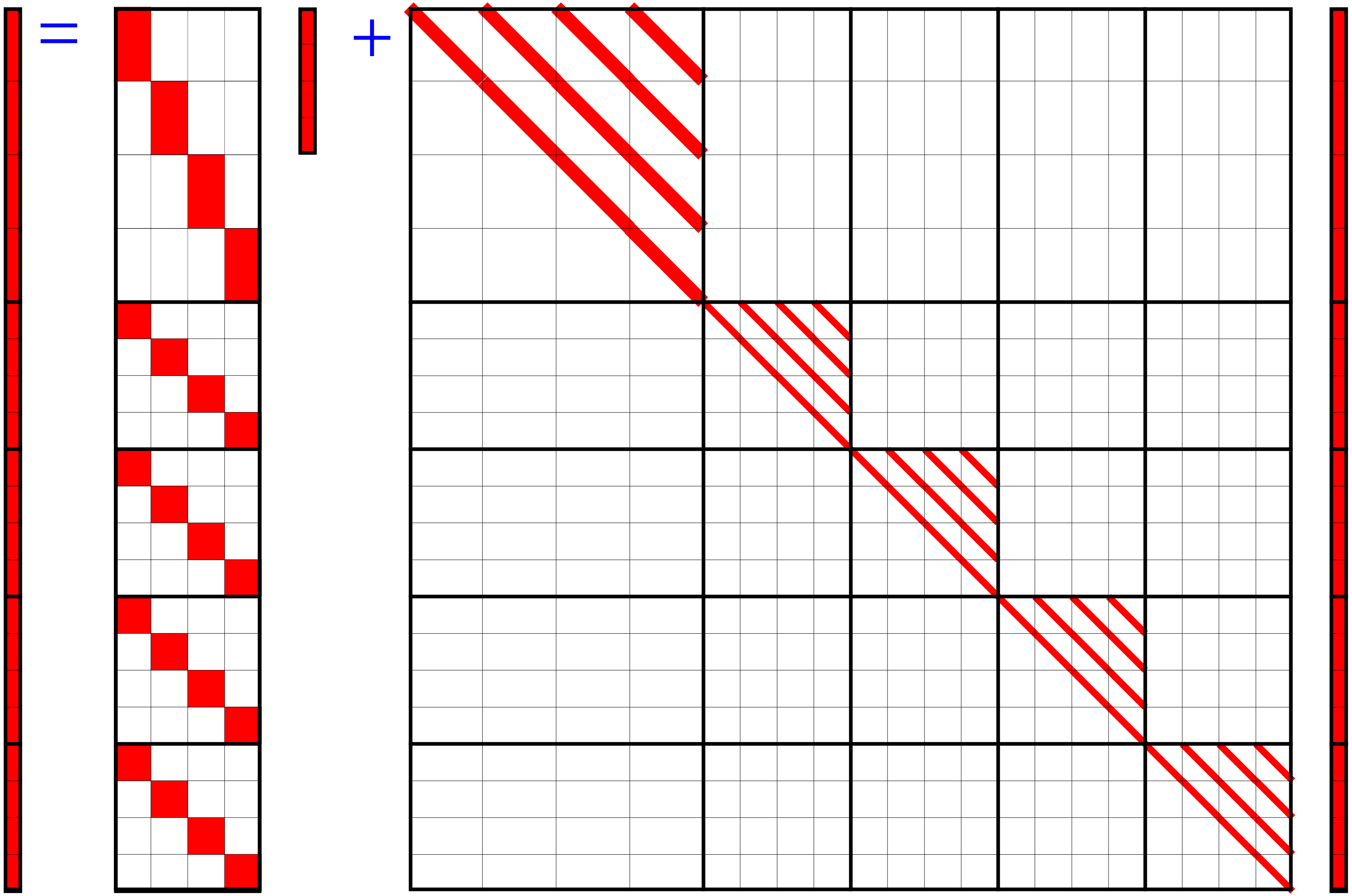
- $T_1 \geq \max(k_1, \ldots, k_G)$ and $\mathbf{vec}(\{\mathrm{vec}(U^{(i)})\}) \sim \big(0, \oplus_i(\Sigma \otimes I_{T_i})\big)$

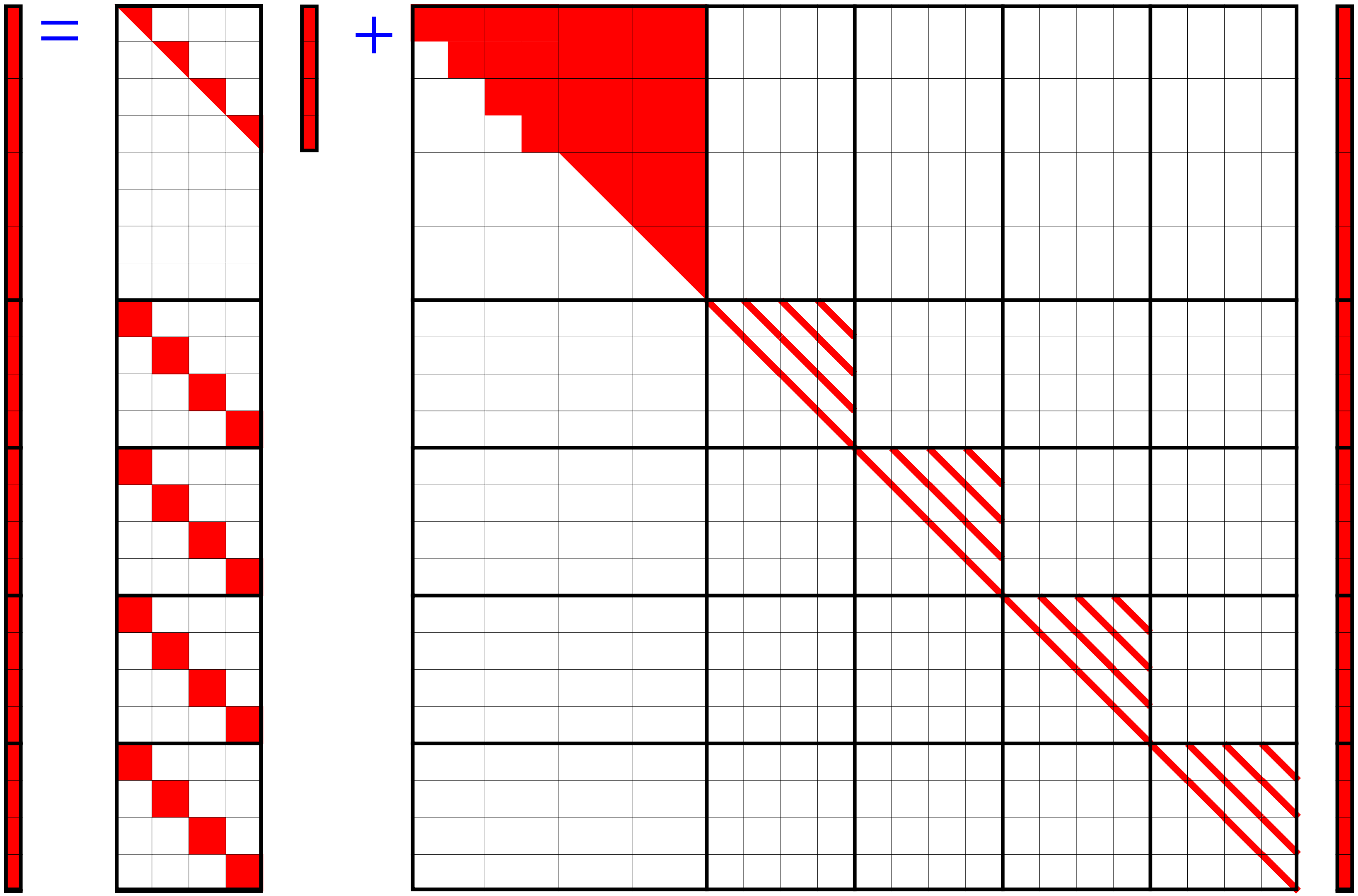
- The SUR model can be written as

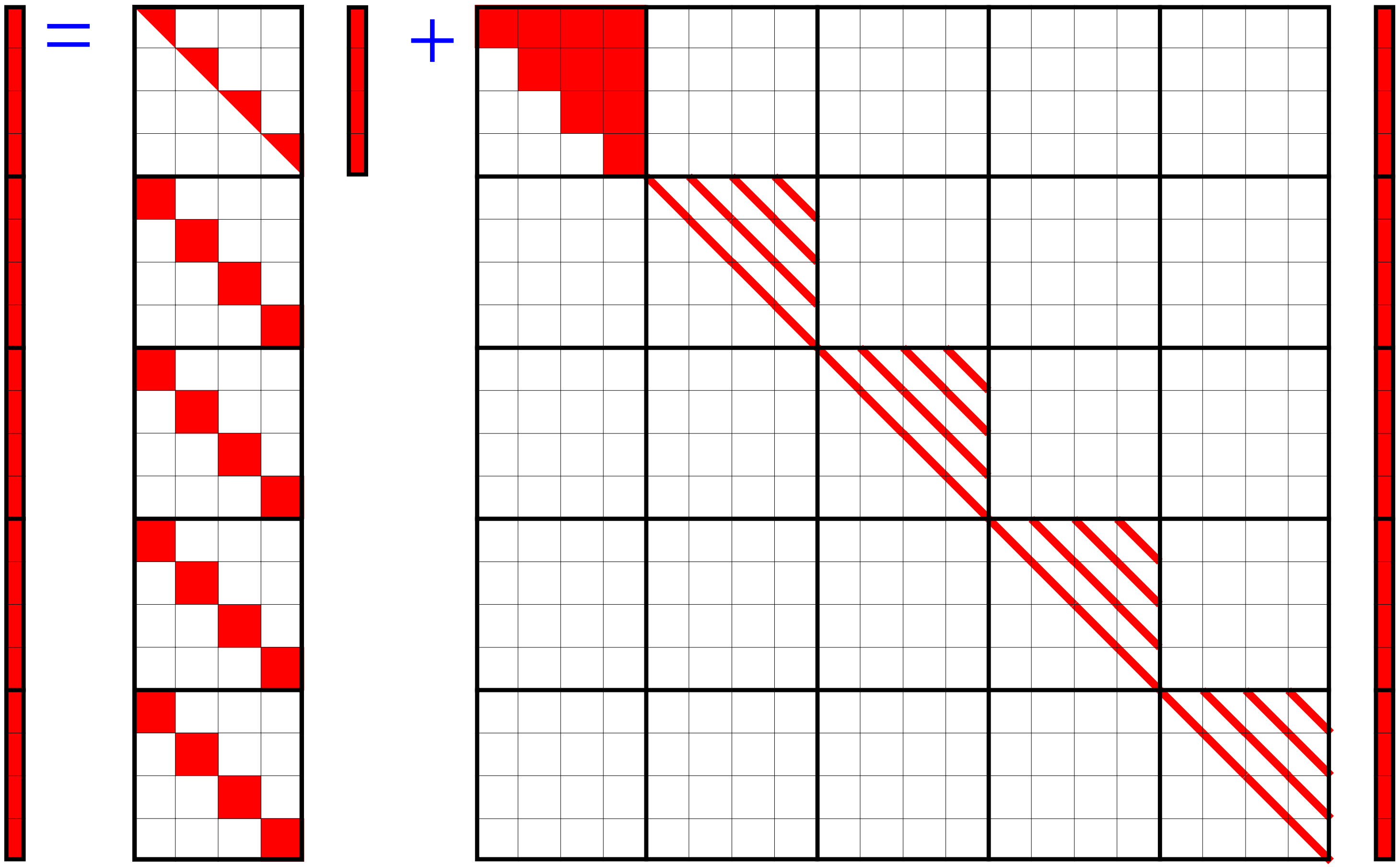
$$\begin{pmatrix} \mathrm{vec}(Y^{(1)}) \\ \vdots \\ \mathrm{vec}(Y^{(G)}) \end{pmatrix} = \begin{pmatrix} \oplus_i X_i^{(1)} \\ \vdots \\ \oplus_i X_i^{(G)} \end{pmatrix} \mathbf{vec}(\{\beta_i\}) + \begin{pmatrix} \mathrm{vec}(U^{(1)}) \\ \vdots \\ \mathrm{vec}(U^{(G)}) \end{pmatrix}$$

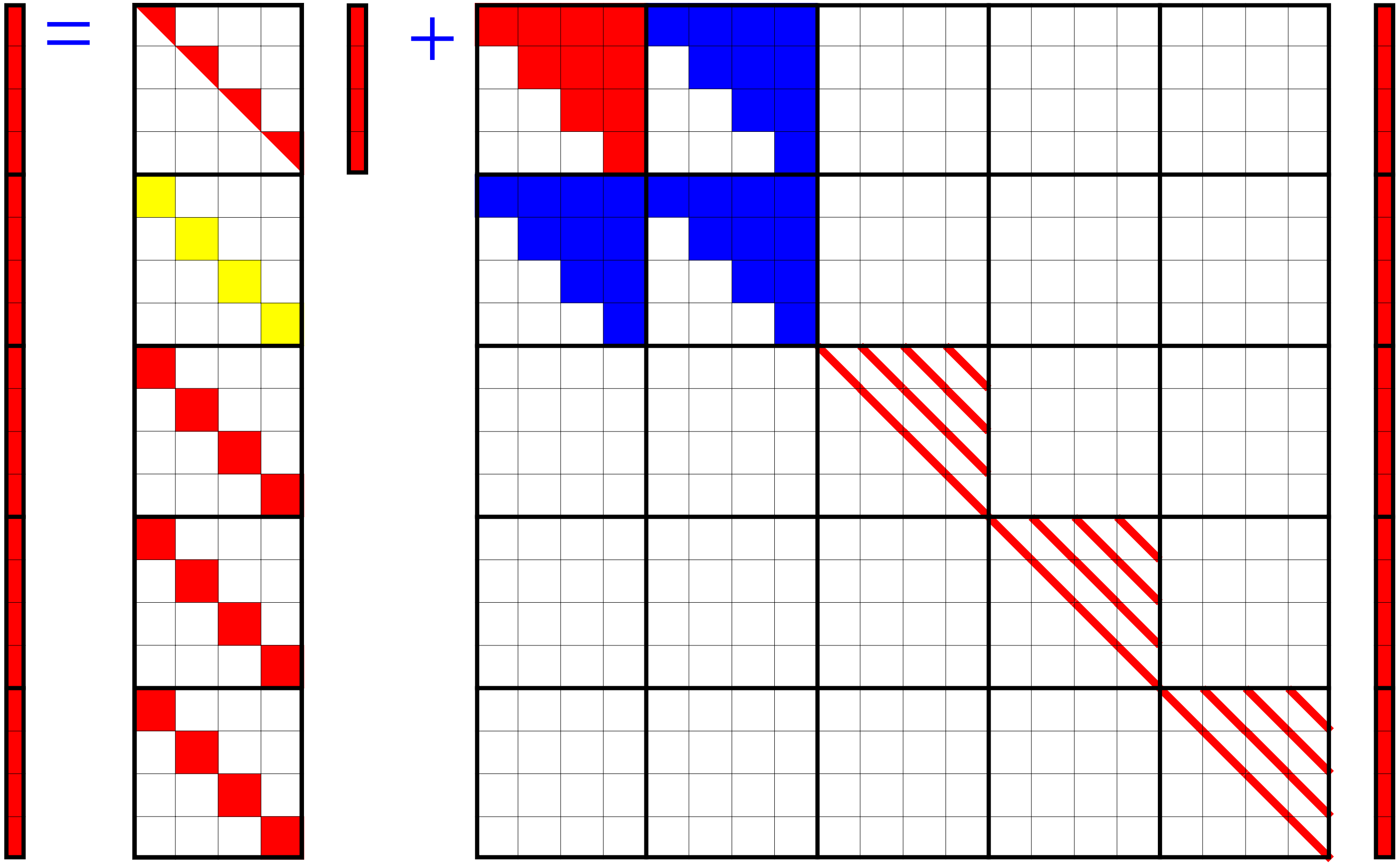
- Solve the SUR model $\mathrm{vec}(Y^{(1)}) = \big(\oplus_i X_i^{(1)}\big)\mathbf{vec}(\{\beta_i\}) + \mathrm{vec}(U^{(1)})$

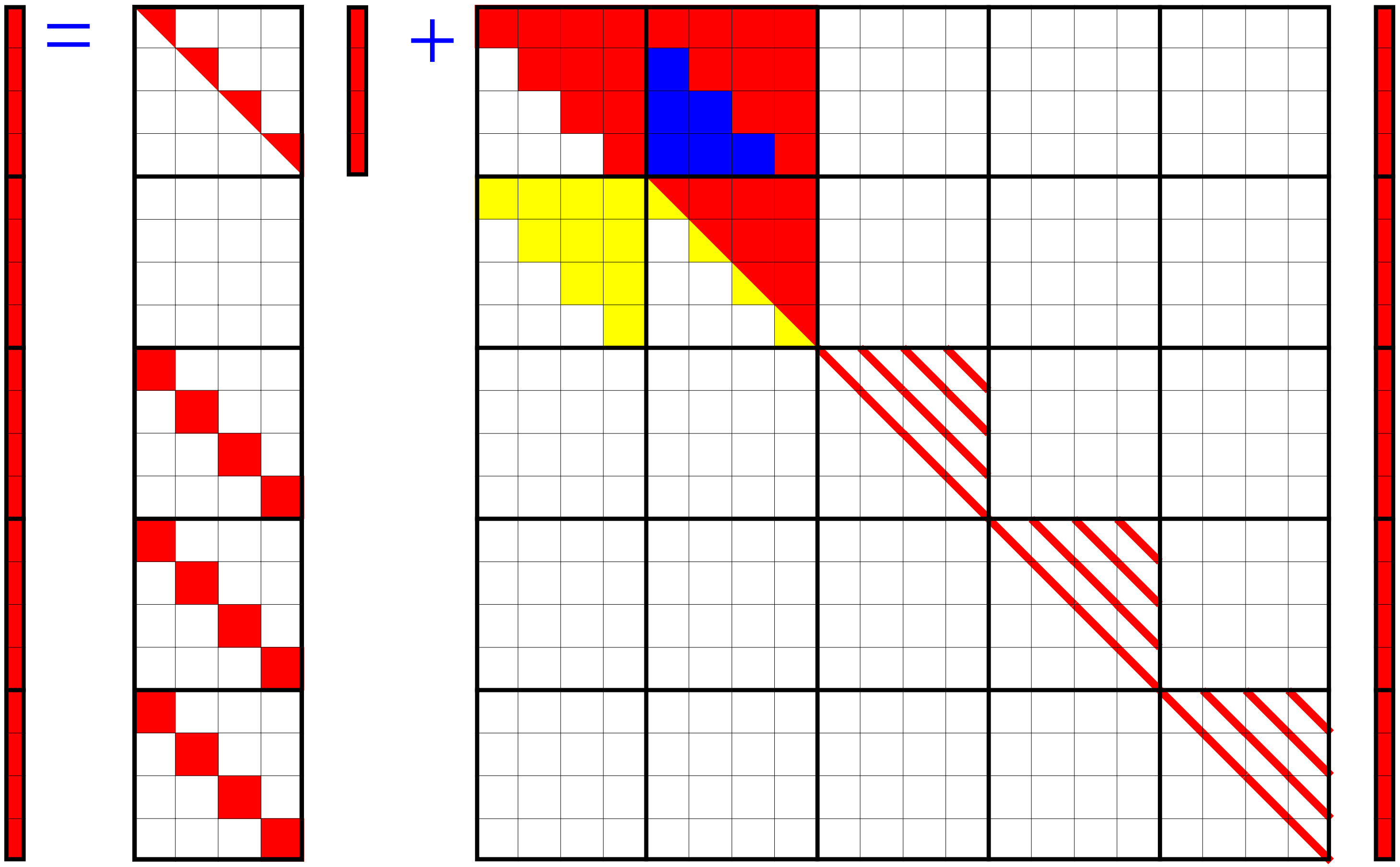
- Update by $\mathrm{vec}(Y^{(j)}) = \big(\oplus_i X_i^{(j)}\big)\mathbf{vec}(\{\beta_i\}) + \mathrm{vec}(U^{(j)})$, for $j = 2, \ldots, s$

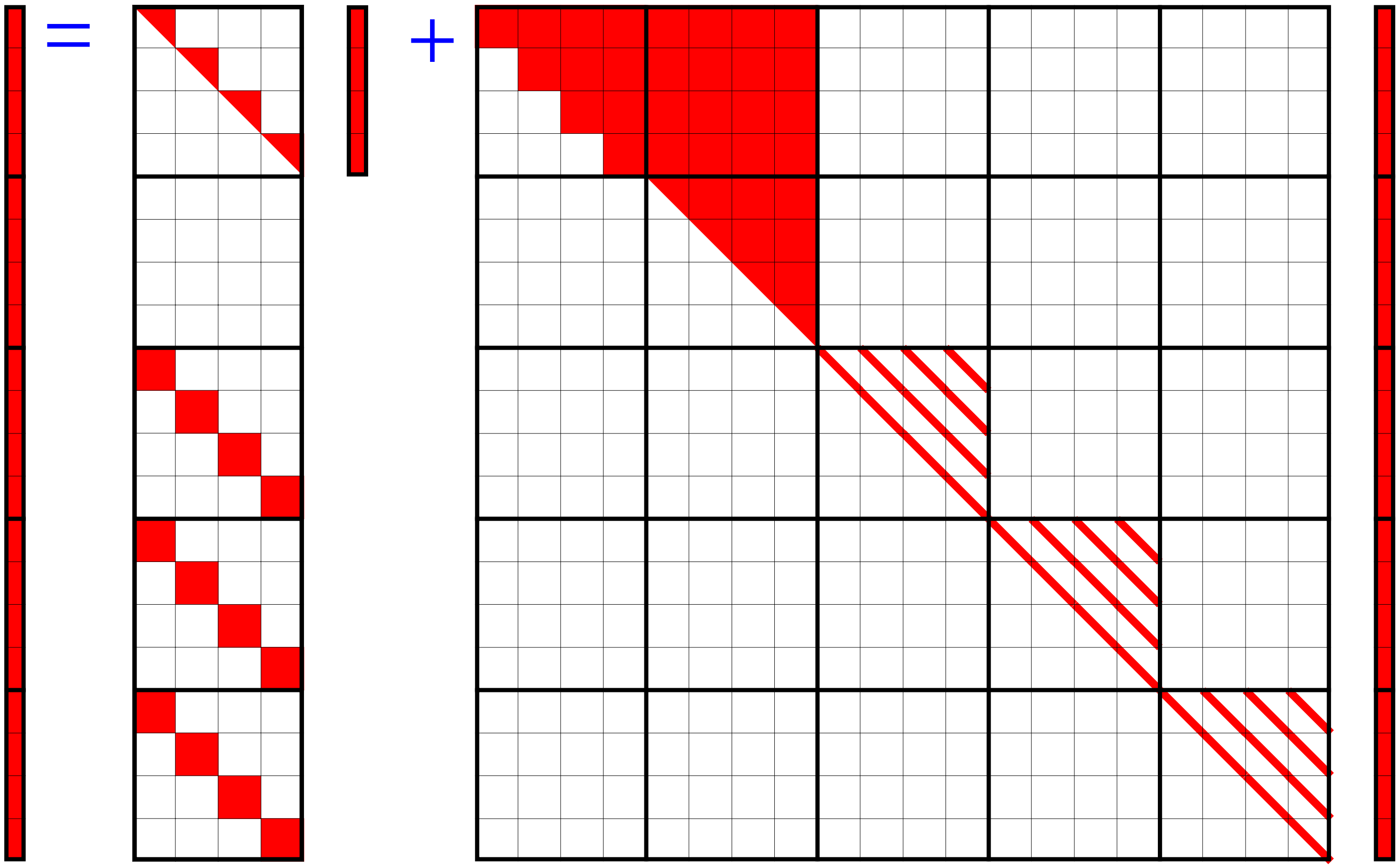


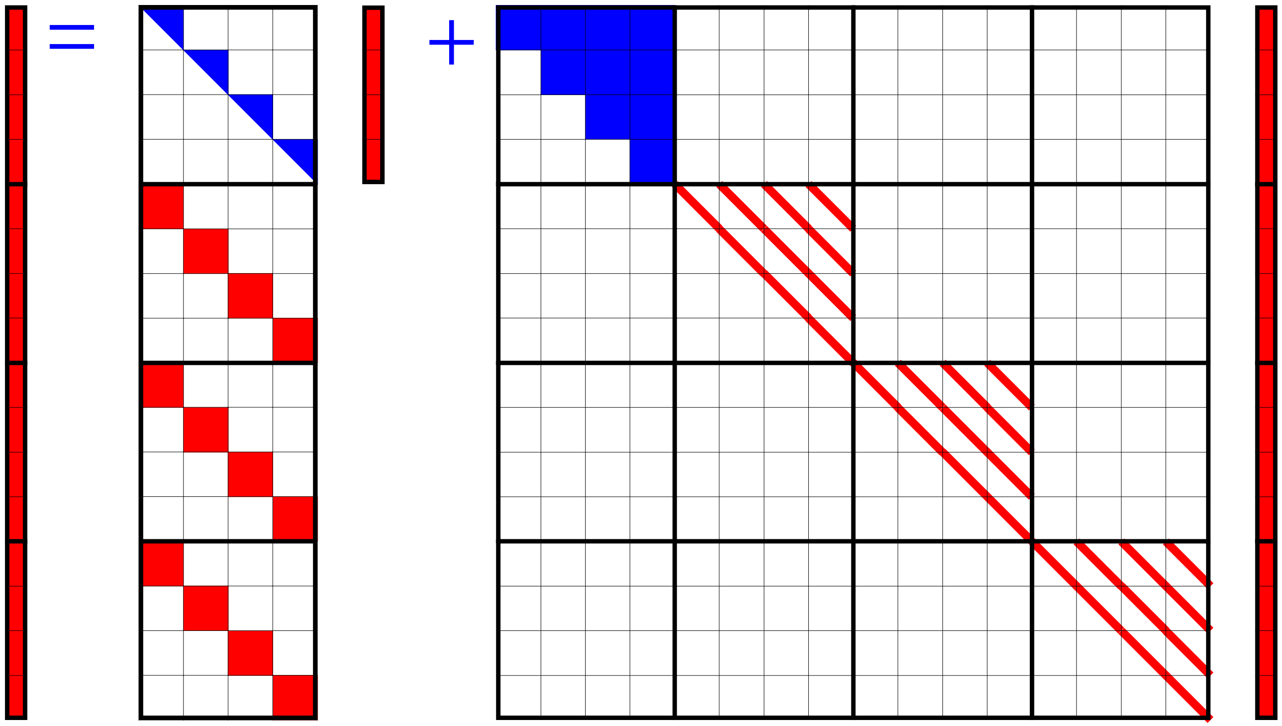












Size Reduction of Large–Scale SUR Models

- Let $X = \begin{pmatrix} X_1 \cdots X_G \end{pmatrix} \mathbb{R}^{T\times K}$, $K = \sum_{i=1}^{G} k_i$ and $T > K$.

- $\tilde{Q}^T X = \begin{pmatrix} \tilde{R} \\ 0 \end{pmatrix}$, $\tilde{R} = \overset{\begin{matrix} k_1 & k_2 & & k_G \end{matrix}}{\begin{pmatrix} \tilde{R}_1 & \tilde{R}_2 & \cdots & \tilde{R}_G \end{pmatrix}} K$ and $\tilde{Q}^T = \begin{pmatrix} \tilde{Q}_1^T \\ \tilde{Q}_2^T \end{pmatrix}$.

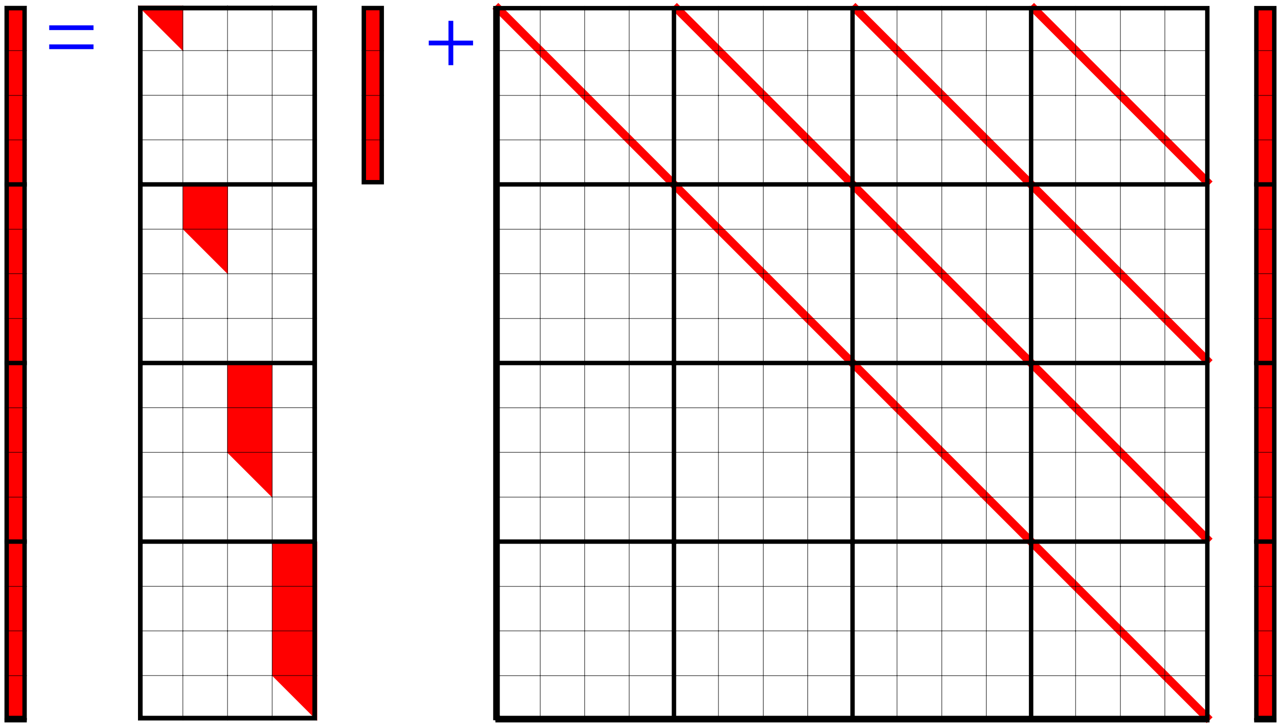
- Pre-multiply the SUR model by $\hat{Q}^T$ where $\hat{Q} = \begin{pmatrix} I_G \otimes \tilde{Q}_1^T & I_G \otimes \tilde{Q}_2^T \end{pmatrix}$.

- $$\begin{pmatrix} \text{vec}(\tilde{Q}_1^T Y) \\ \text{vec}(\tilde{Q}_2^T Y) \end{pmatrix} = \begin{pmatrix} \oplus_i \tilde{R}_i \\ 0 \end{pmatrix} \text{vec}(\{\beta_i\}) + \begin{pmatrix} \text{vec}(\tilde{Q}_1^T U) \\ \text{vec}(\tilde{Q}_2^T U) \end{pmatrix}.$$

- $$\begin{pmatrix} \text{vec}(\tilde{Q}_1^T U) \\ \text{vec}(\tilde{Q}_2^T U) \end{pmatrix} \sim \left(0, \begin{pmatrix} \Sigma \otimes I_K & 0 \\ 0 & \Sigma \otimes I_{T-K} \end{pmatrix}\right).$$

- Reduced-size SUR model: $\text{vec}(\tilde{Q}_1^T Y) = (\oplus_i \tilde{R}_i)\text{vec}(\{\beta_i\}) + \text{vec}(\tilde{Q}_1^T U)$.

Reduced Size SUR Model



- The structure of the regressors should be exploited

Computational results ($\mathbf{G = 10}$)

		OLM algorithms		GLLSP algorithms				Ratio
T	k_i	Lapack	QRD	Lapack	GQRD	Interl.	Recursive	GLLSP/OLM
100	5	0.0227	0.0121	2.9766	0.3384	0.2963	0.1122	9.27
	10	0.1208	0.0327	3.1151	0.4663	0.4511	0.1897	5.80
	15	0.1951	0.0726	3.4684	0.6214	0.5865	0.3085	4.25
	20	0.1979	0.0989	3.6348	0.7296	0.7224	0.4416	4.46
	30	0.3181	0.2138	4.5518	1.0656	1.0375	0.8022	3.75
400	5	0.1812	0.0499	185.3506	8.3259	6.0854	0.4527	9.07
	10	0.6433	0.1731	191.6706	13.1758	9.2547	0.7791	4.50
	15	1.0295	0.3869	198.3687	18.3041	13.0657	1.2969	3.35
	20	1.1602	0.7879	204.5498	23.9272	17.4742	2.0352	2.71
	30	1.8623	1.8808	200.3455	34.8307	25.7010	3.8604	2.07

Computational results ($\mathbf{k_1 = \cdots = k_G = 5}$)

		OLM algorithms		GLLSP algorithms				Ratio
T	G	Lapack	QRD	Lapack	GQRD	Interl.	Recursive	GLLSP/OLM
100	5	0.0014	0.0022	0.5553	0.0507	0.0409	0.0270	19.28
	10	0.0213	0.0119	2.8841	0.3215	0.2906	0.1091	9.17
	15	0.1135	0.0326	10.4153	1.0539	0.9871	0.2853	8.75
	20	0.3039	0.0687	25.5424	2.6647	2.4102	0.5697	8.29
	30	0.7402	0.2122	81.9765	9.4783	8.9455	1.5567	7.34
400	5	0.0162	0.0093	25.6388	1.2633	0.7686	0.1092	11.74
	10	0.1675	0.0498	183.3260	8.2393	6.0076	0.4480	9.00
	15	0.5750	0.1598	586.3929	28.0438	22.2265	1.1694	7.32
	20	1.3203	0.4207	- - -	- - -	- - -	2.4639	5.86
	30	- - -	1.5331	- - -	- - -	- - -	6.6609	4.34

Computational results – Reduced Size model ($G = 10$)

		OLM algorithms		GLLSP algorithms				Ratio
T	k_i	Lapack	QRD	Lapack	GQRD	Interl.	Recursive	GLLSP/OLM
100	5	0.0052	0.0081	0.4015	0.0881	0.0601	0.0477	9.17
	10	0.1088	0.0366	3.0925	0.4421	0.3805	0.1672	4.57
400	5	0.0093	0.0111	0.4249	0.0873	0.0643	0.0491	5.28
	10	0.1236	0.0556	3.1201	0.4482	0.5142	0.1872	3.37
	15	0.3700	0.1443	10.9318	1.2860	1.2046	0.4675	3.24
	20	0.5482	0.3031	27.2676	3.5127	2.9855	0.9567	3.16
	30	1.3749	1.2734	87.4721	13.9188	11.3183	2.7661	2.17

Computational results – Reduced Size model ($k_1 = \cdots = k_G = 5$)

T	G	OLM algorithms Lapack	QRD	GLLSP algorithms Lapack	GQRD	Interl.	Recursive	Ratio GLLSP/OLM
100	5	0.0007	0.0012	0.0055	0.0049	0.0034	0.0050	4.86
	10	0.0068	0.0079	0.3980	0.0834	0.0608	0.0463	6.81
	15	0.0665	0.0277	4.1706	0.5426	0.5045	0.1887	6.81
	20	0.2908	0.0729	25.1177	2.3531	2.1953	0.5194	7.21
400	5	0.0015	0.0019	0.0059	0.0056	0.0040	0.0059	3.93
	10	0.0088	0.0110	0.3973	0.0866	0.0637	0.0496	5.64
	15	0.0746	0.0396	4.3138	0.5676	0.5066	0.1987	5.02
	20	0.3059	0.1010	26.2893	2.5442	2.3426	0.5566	5.51
	30	1.1623	0.4447	79.9213	23.6311	21.8987	2.3073	5.19

Applications

- Multivariate Regression models (i.e. VAR models) with zero coefficient restrictions

$$Y = XB + U, \qquad \text{vec}(U) \sim (0, \Sigma \otimes I), \qquad b_{ij} = 0 \;\text{ for } (i,j) \in \mathcal{S}$$

which can be formulated as the GLLSP

$$\begin{aligned} &\underset{V,B}{\text{argmin}} \|V\|_F \\ &\text{s.t.} \quad Y = XB + VC^T \\ &\qquad\; b_{ij} = 0, \qquad \text{for } (i,j) \in \mathcal{S} \end{aligned}$$

- Simultaneous Equations model:

$$y_i = X_i\beta_i + Y_i\gamma_i + u_i, \qquad i = 1, \ldots, G, \qquad \text{vec}(\{u_i\}) \sim (0, \Sigma \otimes I)$$

Conclusions

- Efficient numerical methods can make feasible the solution of large-scale models.
- Existing numerical libraries cannot be used directly to solve structured problems.
- Efficient algorithms which exploit the structure of the matrices need to be developed.
- Parallelism need to be considered.
- Other interesting matrix problems exists in Econometrics.